MATH3280A Introductory Probability, 2014-2015
Solutions to HW7

P.509 Ex.14

Solution
Let X_i be the random variable of the IQ of the i-th student of the sample, for each $i = 1, 2, ..., n$. $\sigma^2 = Var(X_i) = 170$.

Let $\overline{X}_n = \dfrac{X_1 + X_2 + ... + X_n}{n}$.

We have to find n s.t.

$$P\left(-0.2 \le \overline{X}_n - \mu \le 0.2\right) = 0.98$$

.
The Central Limit Theorem (CLT) states that for all $t \in \mathbb{R}$,

$$\lim_n P\left(\frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \le t\right) = \Phi(t)$$

.
Then we have

$$\begin{aligned}
&P\left(-0.2 \le \overline{X}_n - \mu \le 0.2\right) \\
&= P\left(\overline{X}_n - \mu \le 0.2\right) - P\left(\overline{X}_n - \mu \le -0.2\right) \\
&= P\left(\frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \le \frac{0.2\sqrt{n}}{\sigma}\right) - P\left(\frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \le -\frac{0.2\sqrt{n}}{\sigma}\right) \\
&\approx \Phi\left(\frac{0.2\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{0.2\sqrt{n}}{\sigma}\right) && \text{(by CLT)} \\
&= 2\Phi\left(\frac{0.2\sqrt{n}}{\sqrt{170}}\right) - 1
\end{aligned}$$

$$2\Phi\left(\frac{0.2\sqrt{n}}{\sqrt{170}}\right) - 1 = 0.98 \qquad \text{iff} \qquad \Phi\left(\frac{0.2\sqrt{n}}{\sqrt{170}}\right) = 0.99.$$

From the standard normal table, we have

$$\frac{0.2\sqrt{n}}{\sqrt{170}} \approx 2.33 \qquad \text{i.e.} \qquad n \approx 23073.$$

□

P.510 Ex.20

Solution

Let $X = \sum_{i=1}^{26} X_i$, $\mu = E(X)$, $\sigma^2 = Var(X)$.

$\mu = E(X) = \sum_{i=1}^{26} E(X_i) = 26\left(\frac{26}{51}\right) \approx 13.2549$

Chebyshev's inequality states that for all $t \in \mathbb{R}$,

$$P\left(|X-\mu| \geq t\right) \leq \frac{\sigma^2}{t^2}.$$

Then

$$\begin{aligned}
&P(X \leq 10) \\
&= P(X-\mu \leq 10-\mu) \\
&\leq P(X-\mu \leq -t) + P(X-\mu \geq t) \qquad\qquad \text{where } t = \mu - 10 \\
&= P(|X-\mu| \geq t) \\
&\leq \frac{\sigma^2}{t^2}
\end{aligned}$$

We now compute σ^2.

$$\begin{aligned}
\sigma^2 &= Var(X) \\
&= Var\left(\sum_{i=1}^{26} X_i\right) \\
&= \sum_{i=1}^{26} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)
\end{aligned}$$

$$\begin{aligned} Var(X_i) &= E(X_i^2) - E(X_i)^2 \\ &= \frac{26}{51} - \left(\frac{26}{51}\right)^2 \end{aligned}$$

$$\begin{aligned} Cov(X_i, X_j) &= E((X_i - E(X_i))(X_j - E(X_j))) \\ &= E(X_i X_j - E(X_i) X_j - E(X_j) X_i + E(X_i) E(X_j)) \\ &= E(X_i X_j) - E(X_i)^2 \\ &= P(X_i = 1, X_j = 1) - E(X_i)^2 \\ &= P(X_i = 1) P(X_j = 1 | X_i = 1) - E(X_i)^2 \\ &= \frac{26}{51} \cdot \frac{25}{49} - \left(\frac{26}{51}\right)^2 \end{aligned}$$

Then

$$\begin{aligned} \sigma^2 &= \sum_{i=1}^{26} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \\ &= 26 \left(\frac{26}{51} - \left(\frac{26}{51}\right)^2 \right) + 25 \cdot 26 \left(\frac{26}{51} \cdot \frac{25}{49} - \left(\frac{26}{51}\right)^2 \right) \\ &\approx 6.6301 \end{aligned}$$

Hence

$$P(X \leq 10) \leq \frac{\sigma^2}{t^2} \approx 0.6258$$

□